MA1 - zkouška 26.1.21 - odpoledne

① $$\int \left(\frac{\sin x \cdot \cos x}{1+\sin^4 x} + \frac{2x-5}{2x\,(x+4\sqrt{x}+5)} \right) dx = I_1 + I_2$$

(i) existence: je-li f spojitá v (a,b), pak má v (a,b) funkci primitivní:

v I_1: $\frac{\sin x \cos x}{1+\sin^4 x}$ je spoj. v $\mathbb{R}$

v I_2 $\frac{2x-5}{2x(x+4\sqrt{x}+5)}$ je spojitá v $(0,+\infty)$

$\Big\} \Rightarrow$ integrál zadaný existuje v $(0,+\infty)$ interval 0,5b

(ii) I_1 3b $$\int \frac{\sin x \cos x}{1+\sin^4 x}\,dx \underset{1VS}{=} \left| \begin{matrix} \sin = t \\ \cos x\,dx = dt \end{matrix} \right| = \int \frac{t}{1+t^4}\,dt \underset{1VS}{=} \left| \begin{matrix} t^2 = y \\ 2t\,dt = dy \end{matrix} \right|$$

1b 1,5b

$$= \frac{1}{2}\int \frac{1}{1+y^2}\,dy = \frac{1}{2}\,\mathrm{arctg}\,(y) + C = \frac{1}{2}\,\mathrm{arctg}\,(\sin^2 x) + C$$

(nebo „pedrecr" 1VS $\left| \begin{matrix} \sin^2 x = t \\ 2\sin x \cos x\,dx = dt \end{matrix} \right| = \frac{1}{2}\int \frac{dt}{1+t^2} = \dots$)

(iii) I_2 6,5b $$\int \frac{2x-5}{2x(x+4\sqrt{x}+5)}\,dx \underset{2VS}{=} \left| \begin{matrix} \sqrt{x} = t \\ x = t^2 \\ dx = 2t\,dt \end{matrix} \right| = \int \frac{(2t^2-5)\cdot \cancel{2t}}{\cancel{2t^2}(t^2+4t+5)}\,dt =$$

2b subst.

$$= -\int \frac{1}{t}\,dt + \int \frac{3t+4}{t^2+4t+5}\,dt = -\int \frac{1}{t}\,dt + \frac{3}{2}\int \frac{2t+4}{t^2+4t+5}\,dt - 2\int \frac{1}{(t+2)^2+1}\,dt$$

$$= -\ln t + \frac{3}{2}\ln(t^2+4t+5) - 2\,\mathrm{arctg}\,(t+2) + C = (t = \sqrt{x}) =$$

$$= -\ln\sqrt{x} + \frac{3}{2}\ln(x+4\sqrt{x}+5) - 2\,\mathrm{arctg}\,(\sqrt{x}+2) + C$$

0,5b 2b 2,5b integrace

Rozklad: $$\frac{2t^2-5}{t(t^2+4t+5)} = \frac{A}{t} + \frac{Bt+C}{t^2+4t+5}$$ 2b rozklad

$$2t^2-5 = A(t^2+4t+5) + Bt^2 + Ct, \;\; \forall t$$

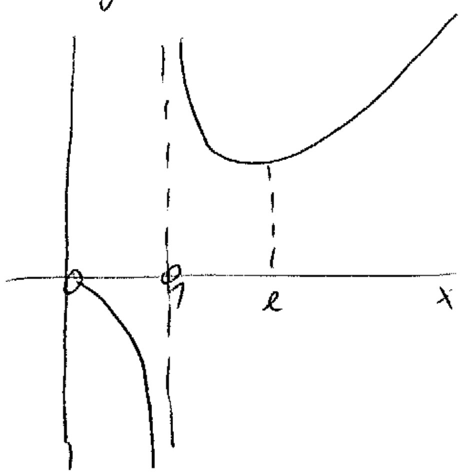
u t^2: $A + B = 2 \Rightarrow B = 3$

u t: $4A + C = 0 \Rightarrow C = 4$

u t^0: $5A = -5 \Rightarrow A = -1$

② $f(x) = \dfrac{x}{\ln x}$:

odhad grafu:



a) $D_f = \{x>0;\ x\neq 1\} = (0,1)\cup(1,+\infty)$
f je spojitá v D_f, $f(x)>0$ v $(1,+\infty)$, $\Big\}$ 1b
$f(x)<0$, v $(0,1)$

$\lim\limits_{x\to 0+} \dfrac{x}{\ln x} = „\dfrac{0}{-\infty}" = 0$, $\lim\limits_{x\to 1\pm} \dfrac{x}{\ln x} = \dfrac{1}{0\pm} = \pm\infty$, 1b 0,5b

$\lim\limits_{x\to +\infty} \dfrac{x}{\ln x} = „\dfrac{\infty}{\infty}" \overset{l'H}{=} \lim\limits_{x\to\infty} \dfrac{1}{\frac{1}{x}} = \infty$ 0,5b

b) $f'(x) = \dfrac{\ln x - x\cdot\frac{1}{x}}{\ln^2 x} = \dfrac{\ln x - 1}{\ln^2 x}$ f' 1b

$f'(x)=0 \Leftrightarrow \ln x = 1 \Leftrightarrow x = e$ $\Rightarrow$ v $x=e$ je ostré lokální minimum, $f(e)=e$ 0,5b
glob. maximum ani minimum fce nemá ($\lim \pm\infty$) 0,5b

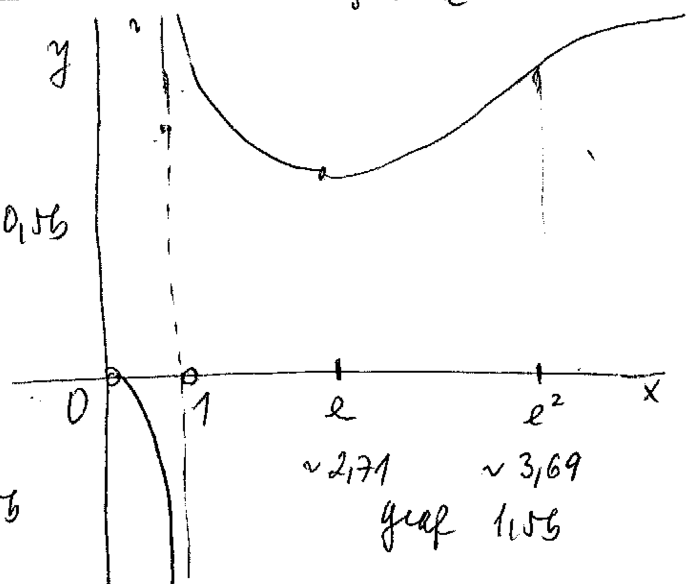
f': 0 − 1 − e +
↘ ↘ ↗ 1b ↗↘

c) $f''(x) = \dfrac{\frac{1}{x}\ln^{\cancel{2}} x - (\ln x - 1)\cdot 2\cancel{\ln x}\cdot\frac{1}{x}}{\cancel{\ln^4 x}\ \ln^3 x} =$ f'' 1,5b

$= \dfrac{1}{x}\ \dfrac{\ln x - (2\ln x - 2)}{\ln^3 x} = \dfrac{1}{x\ln^3 x}(2-\ln x)$, $f''(x)=0 \Leftrightarrow \ln x = 2$, tj. $x = e^2$

− + −
0 ∩ 1 ∪ e^2 ∩ 1b

$\Rightarrow$ v $x=e^2$ je inflexe, $f(e^2) = \dfrac{e^2}{2} \doteq$ 0,5b



~2,71 ~3,69
graf 1,5b

d) asymptoty: $x=1$

šikmá: $\lim\limits_{x\to+\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to\infty} \dfrac{1}{\ln x} = 0$,
(v $x\to+\infty$)
$\Big\}$ 0,5b
ale $b = \lim\limits_{x\to+\infty}(f(x) - a x) = \lim\limits_{x\to\infty} \dfrac{x}{\ln x} = \infty$,

tj. šikmou asymptotu fce nemá (ani „vodorovnou")

(3) $S(\omega)$, kde ω je omezená oblast v rovině, ohraničená grafy $y = x^2 e^x$, $y = x^2$ a přímkou $x = 1$

(i) $x^2 e^x = x^2 \iff x = 0$ tj. $x \in \langle 0,1\rangle$

(ii) $e^x > 1$ pro $x > 0$ $\Rightarrow x^2 e^x \geq x^2$ pro $x \in \langle 0,1\rangle$, tedy (pro $x \geq 0$) 1b

$$\underline{S(\omega)} = \underline{\int_0^1 (x^2 e^x - x^2)\,dx} \; 2b = \int_0^1 x^2 e^x\,dx - \int_0^1 x^2\,dx =$$

= model 3b celkem $= e - 2 - \left[\frac{x^3}{3}\right]_0^1 = \underline{\underline{e - \frac{7}{3}}} \; (>0)$

0,5b

Výpočet $\underline{\int_0^1 x^2 e^x\,dx} \underset{\text{PP}}{=} \left| \begin{array}{ll} u' = e^x & u = e^x \\ v = x^2 & v' = 2x \end{array} \right| = \left[x^2 e^x\right]_0^1 - 2\int_0^1 x e^x\,dx =$

PP 1,5b

$\underset{\text{PP}}{=} \left| \begin{array}{ll} u' = e^x, & u = e^x \\ v = x, & v' = 1 \end{array} \right| = \left[x^2 e^x\right]_0^1 - 2\left(\left[x e^x\right]_0^1 - \int_0^1 e^x\,dx\right) =$

PP 1,5b

$= \left[x^2 e^x - 2x e^x + 2e^x\right]_0^1 = \left[e^x\,(x^2 - 2x + 2)\right]_0^1 =$

$= e \cdot 1 - 2 = \underline{\underline{e - 2}}$ dosazení 0,5b

(4) $y' = \dfrac{x}{\sqrt{1-x^2}}\,(2-y)$, $x \in (-1,1)$

a) (i) $y(x) = 2$, $x \in (-1,1)$ stac. řešení 1b

(ii) $y(x) \neq 2$ – separace: 1b (bez $y \neq 2$ 0,5b)

$$\int \frac{dy}{y-2} = \int \frac{-x}{\sqrt{1-x^2}}\,dx$$ integrace 2b (1+1)

$$\ln|y-2| = \sqrt{1-x^2} + C$$

úprava 1,5b $|y-2| = e^{\sqrt{1-x^2}} \cdot e^{C}$

$$y-2 = e^{\sqrt{1-x^2}} \cdot K \ , \ K \neq 0$$

odstranění abs. hodnoty: 1,5b

$y-2>0 \Rightarrow |y-2| = y-2$ a

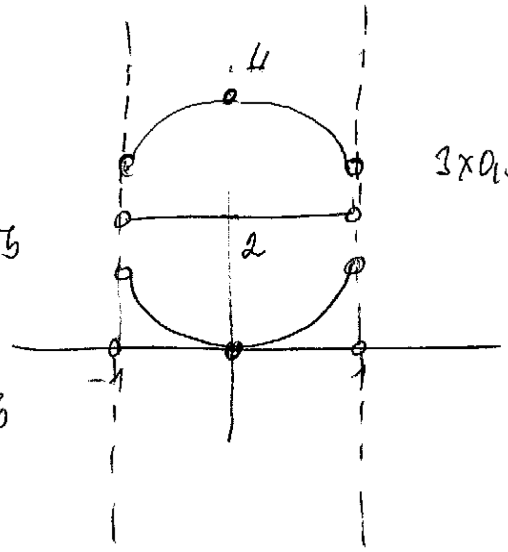
$\underline{y = 2 + e^{\sqrt{1-x^2}} \cdot e^{C}}$

$y-2<0 \Rightarrow |y-2| = -(y-2)$ a

$\underline{y = 2 - e^{\sqrt{1-x^2}} \cdot e^{C}}$

(ii) a (i) : $\underline{y_{OB}(x) = 2 + K e^{\sqrt{1-x^2}}}$, $\underline{K \in R,\ x \in (-1,1)}$

b) počáteční úlohy:



3x0,5b

(i) $y(0) = 4$: $4 = 2 + Ke \Rightarrow \underline{K = 2e^{-1}}$

$\underline{y(x) = 2 + 2e^{\sqrt{1-x^2}-1} \ , \ x \in (-1,1)}$ 0,5b

(ii) $y(0) = 0$: $0 = 2 + Ke \Rightarrow \underline{K = -2e^{-1}}$ 0,5b

$\underline{y(x) = 2 - 2e^{\sqrt{1-x^2}-1} \ , \ x \in (-1,1)}$

(iii) $\underline{y(0) = 2 \quad y(x) = 2}$ – stac. řešení 0,5b

$$\lim_{x\to 1-} y(x) = 2 + K \ , \ \lim_{x\to 1-} f'(x) = \lim_{x\to 1-} K e^{\sqrt{1-x^2}} \cdot \frac{-2x}{2\sqrt{1-x^2}} = -\infty$$

a $y(x)$ je funkce sudá

① (i) $A\cdot B = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, tj. odkud

$\Rightarrow$ (ii) $A\cdot B = I \Rightarrow B = A^{-1}$

(iii) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = B\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, tj. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ 4 \end{pmatrix}$

② $f(x) = \frac{1-\cos x}{x}$, $x \neq 0$; $f(0)=0$

(i) f je spojitá v $x=0$: $\lim\limits_{x\to 0} \frac{1-\cos x}{x} = \text{„}\frac{0}{0}\text{"} \overset{l'H}{=} \lim\limits_{x\to 0} \frac{\sin x}{1} = 0 = f(0)$

(ii) $\underline{f'(0)} = \lim\limits_{x\to 0} \frac{f(x)-f(0)}{x} = \lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \text{„}\frac{0}{0}\text{"} \overset{l'H}{=} \lim\limits_{x\to 0} \frac{+\sin x}{2x} = \underline{\frac{1}{2}}$

(nebo výpočet limit: $\lim\limits_{x\to 0} \frac{1-\cos x}{x} = \lim\limits_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} = \lim\limits_{x\to 0} \underbrace{\frac{\sin x}{x}}_{\to 1} \cdot \underbrace{\sin x}_{\to 0} \cdot \underbrace{\frac{1}{1+\cos x}}_{\to \frac{1}{2}} = 0$

analog. i $f'(0)$)

③ $T_2^0(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2}(x^2)$, $\underline{f(x) = \sqrt{4+\sin(2x)}}$

$\underline{f(0) = \sqrt{4} = 2}$

$\underline{f'(0)} = \frac{1}{\cancel{2}\sqrt{4+\sin(2x)}} \cdot \cos(2x) \cdot \cancel{2} \Big|_{x=0} = \underline{\frac{1}{2}}$

$\underline{f''(x)} = \frac{-\sin(2x)\cdot 2\sqrt{4+\sin(2x)} - \cos(2x)\cdot\frac{1}{\sqrt{4+\sin(2x)}}\cdot\cos 2x}{4+\sin(2x)}\Big|_{x=0} = -\frac{1}{8}$,

tj. $\underline{T_2(x) = 2 + \frac{1}{2}x - \frac{1}{16}x^2}$